Journal of Computational Mathematics, Vol.14, No.1, 1995, 71–97.

DISCRETE APPROXIMATIONS FOR SINGULARLY PERTURBED BOUNDARY VALUE PEOBLEMS WITH PARABOLIC LAYERS, I[*1)]

P.A. Farrell
(*Department of Mathematics and Computer Science, Kent State University, USA*)
P.W. Hemker
(*CWI, Center for Mathematics and Computer Science, Amsterdam, The Netherlands*)
G.I. Shishkin
(*IMM, Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Science, Ekaterinburg, Russia*)

Abstract

In this series of three papers we study singularly perturbed (SP) boundary value problems for equations of elliptic and parabolic type. For small values of the perturbation parameter parabolic boundary and interior layers appear in these problems. If classical discretisation methods are used, the solution of the finite difference scheme and the approximation of the diffusive flux do not converge uniformly with respect to this parameter. Using the method of special, adapted grids, we can construct difference schemes that allow approximation of the solution and the normalised diffusive flux uniformly with respect to the small parameter.

We also consider singularly perturbed boundary value problems for convection-diffusion equations. Also for these problems we construct special finite difference schemes, the solution of which converges ε-uniformly. We study what problems appear, when classical schemes are used for the approximation of the spatial derivatives. We compare the results with those obtained by the adapted approach. Results of numerical experiments are discussed.

In the three papers we first give an introduction on the general problem, and then we consider respectively (i) Problems for SP parabolic equations, for which the solution and the normalised diffusive fluxes are required; (ii) Problems for SP elliptic equations with boundary conditions of Dirichlet, Neumann and Robin type; (iii) Problems for SP parabolic equation with discontinuous boundary conditions.

General Introduction

Consider a substance (or admixture) in a solution with a flux satisfying Fick's law, and with distribution given by a diffusion equation. Let the initial concentration of

* Received March 4, 1995.
1) This work was supported by NWO through grant IB 07-30-012.

the admixture in the material as well as the concentration of the admixture on the boundary of the body be known. It is required to find the distribution of admixture in the material at any given time and also the quantity of admixture (that is the diffusive flux) emitted from the boundaries into the exterior environment. Such problems are of interest in environmental sciences in determining the pollution entering the environment from manufactured sources, such as houses, factories and vehicles, and from industrial and agricultural waste disposal sites, and also in chemical kinetics where the chemical reactions are described by reaction-diffusion equations.

In considering such problems, it is important to note that the diffusion Fourier number, which is given by the diffusion coefficient of the admixture in materials, can be sufficiently small that large variations of concentration occur along the depth of the material. For small values of the Fourier number, diffusion boundary layers appear. Therefore these problems exhibit a singularly perturbed character. The mathematical formulation of such problems have a perturbation parameter which is a small coefficient (the diffusion Fourier number) multiplying the highest derivatives of the differential equation.

Even in the case where only the approximate solution of the singularly perturbed boundary value problem is required, classical numerical methods, such as finite difference schemes and finite element methods[15, 16, 17] exhibit unsatisfactory behaviour. This arises because the accuracy of the approximate solution depends inversely on the perturbation parameter value and thus it deteriorates as the parameter decreases. In [18] it was shown that the use of classical numerical methods does not give approximate solutions with acceptable accuracy even for very fine grids. Thus, even the use of computers with extremely large capacity will not guarantee acceptable accuracy in the answer. To be more precise, it can be shown that the error in the approximate solution on any arbitrarily fine grid is greater than some positive number (independent of the number of grid nodes), for a sufficiently small value of the perturbation parameter (the diffusion Fourier number). For some applied problems such solution accuracy can be satisfactory. However even in these cases dissatisfaction can be caused by the lack of a guarantee than the use of a finer grid will increase the accuracy of the approximation.

More serious problems occur when an accurate approximation of the spatial derivatives of the solution is also required. For example, in order to determine the quantity of admixture which enters the environment per unit of time, it is necessary to compute the gradient of the concentration of the substance along the normal to the surface of the material. When classical finite difference schemes are used it can be expected that errors in the computed diffusive flux will be much larger than those of the computed concentration. Such errors in evaluating fluxes can be often of unacceptable magnitude.

Similar difficulties appear also in problems of heat exchange in cases where the heat Fourier number can take any arbitrary small value. One often requires an accurate approximation of the thermal flux on a boundary of the body.

This series of papers is devoted to the construction of numerical approximations, using finite difference schemes, of singularly perturbed boundary value problems for elliptic and parabolic equations. The simplest example of problems of such type in a one-dimensional case is the problem of a stationary diffusion process with a reacting substance :

$$\varepsilon^2 \frac{d^2}{dx^2} u(x) - c(x)u(x) = f(x), \quad x \in D,$$
$$u(x) = \varphi(x), \quad x \in \Gamma,$$

for $c(x) \geq c_0 > 0$, $x \in \overline{D}$. Here $D = (0,1)$, $\Gamma = \overline{D} \setminus D$ is a boundary of the domain D and the parameter ε can take any value in the interval $(0,1]$. The parameter ε characterises the diffusion coefficient of the substance and the function $c(x)$ characterises the intensity of decay of the diffusion matter. When the parameter tends to zero, diffusion boundary layers appear in a neighbourhood of the boundary.

In the case of regular boundary value problems the error in the approximate solution produced with the use of grid methods, is a function of the smoothness of the solution and of the distribution of the nodes of the grids used. However, the application of classical grid methods for such singularly perturbed boundary value problems leads to loss of accuracy for the approximate solution when the parameter value is small (see, for example, [12, 18] and results in the next section). The following question therefore arise: how to construct and to analyse special numerical methods for solving singularly perturbed boundary value problems, the approximate solution of which converges uniformly with respect to the parameter ε (or, in short, ε-uniformly). The error of the approximate solution obtained by such methods, should be independent of the parameter value and defined only by the number of nodes of the grid used.

Detailed analytic investigations of such special numerical methods dates back to the end of 1960s (see, for example, [3, 12]). These first strong results for problems with boundary layers belong to two different approaches which are used for construction of special numerical methods:

(a) fitted methods[12] on meshes with arbitrary distribution of nodes (for example, on a uniform mesh) the coefficients of difference equations (difference approximations) are chosen (fitted) to ensure parameter-uniform accuracy of the approximate solution; or

(b) methods on special condensing grids (or adaptive meshes)[3]. Those methods use the standard classical difference equations but the nodes of the mesh are redistributed (or adapted, or condensed in the boundary layer) such that parameter-uniform convergence is achieved.

Special, fitted schemes (that is the first approach) are attractive, since they allow the use of meshes with an arbitrary distribution of nodes, e.g. uniform grids (see, for example, [1, 2, 4, 6, 12]). Using the second approach, adapted meshes with classical finite difference approximations, parameter-uniformly convergent schemes were also constructed for a series of boundary value problems (see, for example, [23] and references therein). For some boundary value problems parameter-uniformly convergent schemes were constructed using either the first or the second approach for the same problem (see, for example [6, 20]), or using both approaches together for the same problem (see, for example [18, 19], where different approaches were used in different coordinate directions). In [14] both approaches were used at the same time (a fitted scheme on a grids with condensing nodes in the boundary layer). Thus there is a large variety of special approaches tailored to individual boundary value problems in the literature.

In the case of singularly perturbed boundary value problems, for which accurate estimates of the diffusive fluxes are required, methods must be evolved which approximate both the solution and the normalised fluxes accurately. Investigations of such methods have been sparse in the literature (see, for example, [18]).

In the present paper we consider singular perturbed elliptic and parabolic equations with parabolic boundary layers. For boundary value problems we construct special difference schemes, solutions of which converge ε-uniformly in an ℓ^{∞}- norm. Also approximations of the normalised diffusive fluxes which converge ε-uniformly, are proposed.

In the next section it is shown that the computed solution, for a singularly perturbed ordinary differential equation, which is found using a classical scheme does not converge ε-uniformly. We then consider the construction of special schemes which are ε-uniformly convergent. Grid approximations of solutions and diffusive fluxes for singularly perturbed parabolic equations are considered in this first paper. Approximations of elliptic equation with mixed boundary condition, which admit Dirichlet and Neumann conditions are studied in the second paper. To construct the special schemes in the first two papers methods based on special condensed grids are used.

In the third and last paper we investigate singularly perturbed boundary value problems with discontinuous boundary conditions. In this case fitted methods are used.

The improved special finite difference schemes which allow accurate approximation of both the solutions and the normalised diffusive fluxes for boundary value problems can be effectively applied for the solution and numerical analysis of applied problems with boundary and interior layers. The methods for construction of special schemes developed here can also be used to construct and investigate special schemes for more general singularly perturbed boundary value problems (see, for example, [7, 8, 23]).

The necessity to construct special schemes

In order to demonstrate the problems which may appear in the numerical solution process, we consider the following simple example of a singularly perturbed ordinary differential equation for a boundary value problem:

$$L_{(1.1)}u(x) \equiv \varepsilon^2 \frac{d^2}{dx^2}u(x) - u(x) = -1, \quad x \in D, \tag{1.1a}$$

$$u(0) = u(1) = 0; \quad \varepsilon \in (\,0,1\,], \tag{1.1b}$$

where $D = (0,1)$. For the solution of this problem we should like to use classical numerical methods, for example finite difference schemes[2].

The standard scheme for the problem (1.1) is defined as follows. In the interval $\overline{D}$, we introduce the grid

$$\overline{D}_h = \overline{\omega}_1, \tag{1.2}$$

where $\overline{\omega}_1$ is a uniform grid with a step-size $h = 1/N$, and $N+1$ is the number of nodes of the grid $\overline{\omega}_1$. For the problem (1.1) we employ the classical difference scheme

$$\Lambda_{(1.3)}z(x) \equiv \{\varepsilon^2\delta_{x\overline{x}} - 1\}z(x) = -1, \quad x \in D_h, \tag{1.3}$$
$$z(0) = z(1) = 0.$$

Here $D_h = D \cap \overline{D}_h$, $\delta_{x\overline{x}}z(x)$ is the second order central difference approximation to the second derivative

$$\delta_{x\overline{x}}z(x) = h^{-1}(\delta_x - \delta_{\overline{x}})z(x),$$

$$\delta_x z(x) = h^{-1}(z(x+h) - z(x)), \quad \delta_{\overline{x}}z(x) = h^{-1}(z(x) - z(x-h)).$$

It is known (see, for example, [8, 12]) that the error of the scheme (in the ℓ^∞-norm) depends on the value of the parameter ε and on the grid step-size

$$| \, u(x) - z(x) \, | \leq Q(\varepsilon)h^2, \quad x \in \overline{D}_h. \tag{1.4}$$

Here the constant $Q(\varepsilon)$ essentially depends on the parameter value.

Moreover, for sufficiently small values of the parameter, that is, for $\varepsilon = \varepsilon(h) = h^{-1}$, this error becomes larger than some positive constant [18]

$$\max_{\overline{D}_h} | \, u(x) - z(x) \, | \geq m_{(1.5)} > 0 \quad for \quad h \to 0 \tag{1.5}$$

where $u(x) = u(x;\varepsilon)$, $z(x) = z(x;\varepsilon,h)$. That is, for any very small step-size of the grid and an arbitrary value of the parameter ε, $\varepsilon \in (0,1]$, a value of ε can be found such that the error is not less than a positive constant.

It follows from the estimate (1.4) that the difference scheme (1.3), (1.2) converges as $h \to 0$ (or $N \to \infty$) for a fixed value of the parameter. However, according to the estimate (1.5) this difference scheme does not converge uniformly with respect to the small parameter ε (that is it does not converge ε-uniformly).

It is desirable to have numerical methods, for which the error in the approximate solution tends to zero independently of the parameter ε as $N \to \infty$, that is methods in which the approximate solution converges ε-uniformly to the actual solution for $N \to \infty$.

The importance of this criterion for applications, in particular, in the case of the boundary value problem (1.1) is demonstrated by the following numerical experiments.

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[2] The notation $L_{(j.k)}$, $L^i_{(j.k)}$ (or $f_{(j.k)}(x)$, $f^i_{(j.k)}(x)$) means that these operators (or functions) are first introduced in formula $(j.k)$.

The solution of problem (1.1) is given by the following expression:

$$u(x) = u(x;\varepsilon) = 1 - \frac{e^{-x/\varepsilon} + e^{-(1-x)/\varepsilon}}{1 + e^{-1/\varepsilon}}, \quad x \in \overline{D},$$

Note that the function $u(x)$ satisfies the following relations

$$0 \le u(x) < 1, \quad x \in \overline{D},$$

$$\max_{\overline{D}} u(x) = u(1/2), \quad \lim_{\varepsilon\to 0} u(1/2) = 1.$$

In Table 1 we give the results of computing the value $E(\varepsilon, N)$,

$$E(\varepsilon, N) = \max_{\overline{D}_h} e(x;\varepsilon,N), = \max_{\overline{D}_h} |u(x;\varepsilon) - z(x;\varepsilon,N)|$$

which is the maximum local error on $\overline{D}_h$. Here $u(x;\varepsilon)$ is the solution of problem (1.1), and $z(x;\varepsilon,N)$ the solution of problem (1.3). The values of $\overline{E}(N)$, are also given, where

$$\overline{E}(N) = \max_{\varepsilon} E(\varepsilon, N), \quad \varepsilon = 2^{-12} \ldots, 1.0$$

is the largest error of the approximate solution for a fixed value of N and ε varying over the values shown in Table 1

The value $\overline{E}(N)$ defines the best guaranteed accuracy which is obtained by using the scheme (1.3), (1.2) to solve the problem (1.1) for a given N and various values of the parameter $\varepsilon = 4^{-m}$, $m = 0, 1, \cdots, 6$.

Table 1: Table of errors $E(N, \varepsilon)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024	$u(0.5;\, \varepsilon)$
1	5.29e−04	3.33e−05	2.08e−06	1.30e−07	8.14e−09	1.13e−01
2^{-2}	1.99e−02	1.33e−03	8.34e−05	5.21e−06	3.26e−07	7.34e−01
2^{-4}	3.76e−02	1.41e−02	9.53e−04	5.99e−05	3.74e−06	9.99e−01
2^{-6}	3.88e−03	3.74e−02	1.41e−02	9.53e−04	5.98e−05	1.00e+0 0
2^{-8}	2.44e−04	3.88e−03	3.74e−02	1.41e−02	9.53e−04	1.00e+0 0
2^{-10}	1.53e−05	2.44e−04	3.88e−03	3.74e−02	1.41e−02	1.00e+0 0
2^{-12}	9.54e−07	1.53e−05	2.44e−04	3.88e−03	3.74e−02	1.00e+0 0
$\overline{E}(N)$	3.76e−02	3.74e−02	3.74e−02	3.74e−02	3.74e−02	

It follows from Table 1 that the solution of the difference scheme (1.3), (1.2) converges to the solution of problem (1.1) for a fixed value of the parameter. However, the error behaviour is not regular with increasing N. The error decreases with increasing N only for $N \ge 4\varepsilon^{-1}$. The approximate solution does not converge ε-uniformly. Indeed, for a fixed value of N the largest error is found for the parameter value $\varepsilon = \varepsilon(N) = 4^{-1}N^{-1}$, and this error is equal to $3.74 \cdot 10^{-2}$. For any large value of N we cannot guarantee an accuracy better than $3.74 \cdot 10^{-2}$. For the worst realisable error $\overline{E}(N)$ the lower bound $\overline{E}(N) \ge 3.74 \cdot 10^{-2}$ holds. The relative worst realisable error for a fixed N, namely, the value $\delta(N) \equiv \overline{E}(N)[\ \max_{\overline{D}} |u(x)|\ ]^{-1} \geq \overline{E}(N)$, is independent of N and is equal to 3.74 %.

Although the approximate solution does not converge ε-uniformly, this level of accuracy in the computed solution can be acceptable in some cases . The computed solution gives a good qualitative representation of the exact solution behaviour for all values of the parameter ε.

However the accuracy issue appears more significant in the case where, for problem (1.1), it is required to find the gradient of the function $u(x)$ on a boundary (at the ends of the interval D). The derivative $(d/dx)u(x)$ increases unboundedly on the boundary as the parameter ε tends to zero. However, the value $P(x) \equiv \varepsilon(d/dx)u(x)$ (we call this value the normalised diffusion flux, or more briefly the normalised flux) remains bounded ε-uniformly. Therefore it is natural to consider the following problem:

find for boundary value problem (1.1) the solution $u(x)$, $x \in \overline{D}$ and the normalised diffusion flux $P(x)$ on the boundary Γ. (1.6)

Note that for the function $P(x) = P(x;\varepsilon)$ the relations

$$\max_{\overline{D}} |\, P(x)\, | \leq 1;$$

$$P(0;\varepsilon) = -P(1;\varepsilon) > 0; \quad \lim_{\varepsilon \to 0} P(0;\varepsilon) = 1$$

hold.

To solve the problem (1.1), (1.6) we apply the difference scheme (1.3), (1.2). The value $P(0)$ is approximated by the value

$$P^{h+}(x) \equiv \varepsilon \delta_x z(x), \quad x = 0 \tag{1.7}$$

which is the computed normalised diffusive flux at the point $x = 0$.

In Table 2 we give the results of computing the value $Q(\varepsilon, N)$

$$Q(\varepsilon, N) = |\ P(0) - P^{h+}(0)\ |,$$

which is the error in the normalised flux on the boundary $x = 0$ for various values of ε and N. Values of $\overline{Q}(N)$ are also given where

$$\overline{Q}(N) = \max_{\varepsilon = 4^{-m}, m=0,1,\cdots,6.} Q(\varepsilon, N).$$

The value $\overline{Q}(N)$, which is the best guaranteed accuracy (for varying ϵ) of the computed normalised flux at $x = 0$, that can be obtained when using the scheme (1.3), (1.2), (1.7) to solve the problem (1.1), (1.6) for a given N and various values of the parameter ε.

It follows from Table 2 that the value $P^{h+}(0) = P^{h+}(0;\varepsilon, N)$, the computed normalised flux at $x = 0$, converges to the value $P(0;\varepsilon)$ with increasing N, for a fixed value of the parameter ε. However, they do not converge ε-uniformly. The error $Q(\varepsilon, N)$ remains constant for a constant product ϵN. Moreover, the error $Q(\varepsilon, N)$ tends to the value P_0 with decreasing ε for any fixed N, where

$$P_0 \equiv \lim_{\varepsilon \to 0} P(0;\varepsilon) = 1.$$

Table 2: Table of errors of the normalised flux $Q(N,\varepsilon)$ for the classical scheme

$\varepsilon \backslash N$	4	16	64	256	1024	$P(0;\,\varepsilon)$
1	1.22e−01	3.11e−02	7.80e−03	1.95e−03	4.88e−04	4.62e−0 1
2^{-2}	3.93e−01	1.18e−01	3.08e−02	7.78e−03	1.95e−03	9.64e−0 1
2^{-4}	7.64e−01	3.82e−01	1.17e−01	3.08e−02	7.78e−03	1.00e+00
2^{-6}	9.38e−01	7.64e−01	3.82e−01	1.17e−01	3.08e−02	1.00e+00
2^{-8}	9.84e−01	9.38e−01	7.64e−01	3.82e−01	1.17e−01	1.00e+00
2^{-10}	9.96e−01	9.84e−01	9.38e−01	7.64e−01	3.82e−01	1.00e+00
2^{-12}	9.99e−01	9.96e−01	9.84e−01	9.38e−01	7.64e−01	1.00e+00
$\overline{Q}(N)$	9.99e−01	9.96e−01	9.84e−01	9.38e−01	7.64e−01	

In Table 3 the values $\lambda(\varepsilon, N)$ are given where

$$\lambda(\varepsilon, N) \equiv \frac{P(0;\varepsilon)}{P^{h+}(0;\varepsilon,N)} = \frac{d}{dx}u(0;\varepsilon)\;[\;\delta_x z(0;\varepsilon,N)\;]^{-1}$$

denotes the ratio of the exact and the computed flux at $x = 0$.

Table 3: Table of ratios of the normalised fluxes $\lambda(N,\varepsilon)$ for the classical scheme

$\varepsilon \backslash N$	4	16	64	256	1024	$P(0;\,\varepsilon)$
1	1.36e+00	1.07e+00	1.02e+00	1.00e+00	1.00e+00	4.62e−01
2^{-2}	1.69e+00	1.14e+00	1.03e+00	1.01e+00	1.00e+00	9.64e−01
2^{-4}	4.24e+00	1.62e+00	1.13e+00	1.03e+00	1.01e+00	1.00e+00
2^{-6}	1.61e+01	4.24e+00	1.62e+00	1.13e+00	1.03e+00	1.00e+00
2^{-8}	6.40e+01	1.61e+01	4.24e+00	1.62e+00	1.13e+00	1.00e+00
2^{-10}	2.56e+02	6.40e+01	1.61e+01	4.24e+00	1.62e+00	1.00e+00
2^{-12}	1.02e+03	2.56e+02	6.40e+01	1.61e+01	4.24e+00	1.00e+00
$\overline{\lambda}(N)$	1.02e+03	2.56e+02	6.40e+01	1.61e+01	4.24e+00	

This ratio of the exact derivative to its computed difference approximation on the boundary $x = 0$, increases unboundedly for a fixed value of N as the parameter ε tends to zero. The value also increases very sharply when $N \to \infty$ and $\varepsilon N \to 0$ (that is for $\varepsilon \ll N^{-1}$). In these cases the computed flux gives a value which significantly underestimates the actual derivative. This means that, if the classical difference scheme (1.3), (1.2), (1.7) is used, the normalised flux is not even qualitatively approximated by the computed flux in an ε-uniform sense.

In the case of singularly perturbed elliptic equations, for which reduced equations do not contain spatial derivatives, and the principal term in the singular part of the problem solution is described by an equation similar to (1.1a) (see, for example, [18, 19, 20]), so it can be expected that, when solving such singularly perturbed elliptic and parabolic equations with classical difference schemes, computational problems will arise.

Part I

BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS

1. Introduction

Here we study boundary value problems for singularly perturbed parabolic equations. Using a special condensing grid we construct special difference schemes, which approximate the solution and the normalised flux ε-uniformly. Using numerical examples we compare the classical and the special schemes and we show the effectiveness of the constructed schemes.

For the open interval $D = (0, d)$, on the domain

$$G = D \times (\,0, T\,], \quad S = S(G) = \overline{G} \setminus G, \tag{1.8}$$

we consider a boundary value problem for the parabolic equation

$$L_{(1.9)}u(x,t) \equiv \{\,\varepsilon^2 a(x,t)\frac{\partial^2}{\partial x^2} - c(x,t) - p(x,t)\frac{\partial}{\partial t}\,\}u(x,t) = f(x,t),\ (x,t) \in G, \tag{1.9a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{1.9b}$$

Here the functions $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, and also the function $\varphi(x,t)$ are sufficiently smooth functions on the sets $\overline{G}$ and S respectively. Moreover

$$a_0 \le a(x,t) \le a^0, \quad c(x,t) \ge 0, \quad p(x,t) \ge p_0, \quad (x,t) \in \overline{G},$$

$a_0,\ p_0 > 0$, $\varepsilon \in (\,0, 1]$. Suppose that at the corner points $S^* = \{(0,0)\ ,\ (d,0)\}$ compatibility conditions are satisfied, [13], which ensure smoothness of the solution to the boundary value problem for a fixed value of the parameter. The solution of the boundary value problem is a function $u \in C^{2,1}(G) \cap C^{1,0}(\overline{G})$, which satisfies the equation on G and the boundary condition on S. We wish to find the solution and the derivative $(\partial/\partial x)u(x,t)$, $(x,t) \in \overline{G}$.

When the parameter ε tends to zero, a parabolic boundary layer appears in the neighbourhood of the set S^1, that is the lateral boundary of the set G. Note that the derivative $(\partial/\partial x)u(x,t)$, in the neighbourhood of the boundary layer, increases unboundedly when the parameter tends to zero. It is therefore convenient to consider, instead of the gradient $(\partial/\partial x)u(x,t)$, the value $\varepsilon(\partial/\partial x)u(x,t)$, $(x,t) \in \overline{G}$ which is bounded uniformly with respect to the parameter. The value

$$P(x,t) = \varepsilon\frac{\partial}{\partial x}u(x,t), \quad (x,t) \in \overline{G}$$

is called the normalised diffusive flux. In the case of problem (1.9) it is required to find the functions $u(x,t)$, $P(x,t)$, $(x,t) \in \overline{G}$.

On the set $\overline{G}_{(1.8)}$, we shall also consider the boundary value problem for the quasi-linear parabolic equation

$$L_{(1.10)}(u(x,t)) \equiv \{\,\varepsilon^2 a(x,t)\frac{\partial^2}{\partial x^2} - p(x,t)\frac{\partial}{\partial t}\,\}u(x,t) - \tag{1.10a}$$

$$-g(x,t,u(x,t))=0,\quad (x,t)\in G,$$
$$u(x,t)=\varphi(x,t),\quad (x,t)\in S.$$

The function $g(x,t,u)$ is a sufficiently smooth function on the set $\overline{G}\times R$, satisfying

$$-m_{(1.10)}\le \frac{\partial}{\partial u}g(x,t,u)<\infty,\quad (x,t,u)\in\overline{G}\times R, \tag{1.10b}$$

and the functions $a(x,t)$, $p(x,t)$, $\varphi(x,t)$ and the parameter ε satisfy the conditions above. Again, it is required to find the functions $u(x,t)$ and $P(x,t)$, $(x,t)\in\overline{G}$, that is, the solution of problem (1.10) and normalised flux.

We arrive at problem (1.10) by considering, for example, the diffusion of a substance (e.g. pollution) in a homogeneous layer of solid material of thickness L. When the concentration of substance C depends only on y, that is the distance to the surface, then the distribution of the substance in the layer is described by the diffusion equation

$$D\frac{\partial^2}{\partial y^2}C(y,\tau)-\frac{\partial}{\partial\tau}C(y,\tau)=F(y,\tau),\ 0<y<L,\ 0<\tau<\vartheta.$$

Here D is the diffusion coefficient, and the function $F(y,\tau)$ defines the source. Using variables $x=L^{-1}y$, $t=\vartheta^{-1}\tau$ and denoting $\varepsilon^2=D\vartheta L^{-2}$, $u(x,t)=C(y(x),\tau(t))$, $f(x,t)=\vartheta F(y(x),\tau(t))$, we obtain an equation of the form (1.9a) where $a=p\equiv 1$, $c\equiv 0$. The parameter ε^2 is the diffusion Fourier number $F_O^D=D\vartheta L^{-2}$. The diffusive flux of the substance is defined by the formula

$$D\frac{\partial}{\partial y}C(y,\tau)=DL^{-1}\frac{\partial}{\partial x}u(x,t)=D^{1/2}\vartheta^{-1/2}\varepsilon\frac{\partial}{\partial x}u(x,t)=$$
$$=D^{1/2}\vartheta^{-1/2}P(x,t),$$

where $x=x(y)$, $t=t(\tau)$.

The diffusion coefficients for different media vary considerably, ranging from $10^{-5}\ m^2/sec$ for gases to $10^{-14}\ m^2/sec$ for solid materials. For example, the diffusion coefficient of phenol in air and water is $0.8\cdot 10^{-5}\ m^2/sec$ and $0.8\cdot 10^{-9}\ m^2/sec$ respectively[5]. The diffusion Fourier number is determined by the diffusion coefficient and also by the size of the material samples and by the time period of the diffusion process. For dwellings or local air reservoirs, L is a value between 10 and $10^3\ m$ and ϑ is between 10 min and 1 hour, then F_O^D takes on values between $5\cdot 10^{-9}$ and $3\cdot 10^{-4}$ for the process of phenol diffusion.

For thermal processes the Fourier number is defined by the formula $F_O^T=a\vartheta L^{-2}$, where a is the thermal conductivity of the material. In the case of rapidly varying processes F_O^T becomes a small parameter. It is necessary not only to solve accurately for the temperature but also for the thermal gradients, since physically important variables such as thermal stresses depend directly on them.

We now describe the problems which appear when (1.9) is solved using classical finite difference schemes. For example, consider the boundary value problem

$$L_{(1.11)}u(x,t)\equiv\{\varepsilon^2\frac{\partial^2}{\partial x^2}-1-\frac{\partial}{\partial t}\}u(x,t)=0,\ (x,t)\in G, \tag{1.11a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S,$$

where $d = 1$, $T > 1$, and $\varphi(x,t)$ is a sufficiently smooth function, defined on S, satisfying

$$\left.\begin{array}{l} \varphi(0,t) = 1 \\ \varphi(1,t) = 0 \end{array}\right\} \; 1 < t \leq T. \tag{1.11b}$$

To solve problem (1.11) we use the following classical difference scheme [16]. On the set $\overline{G}$ introduce a rectangular grid

$$\overline{G}_h = \overline{\omega}_1 \times \overline{\omega}_0, \tag{1.12a}$$

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are uniform grids, including the end-points, respectively on the intervals [0,1], [0,T], with step-sizes $h = N^{-1}$, $\tau = N_0^{-1}$, where $N+1$ and N_0+1 are the number of nodes of the grids $\overline{\omega}_1$ and $\overline{\omega}_0$ respectively. For problem (1.11) the difference scheme

$$\Lambda_{(1.12)} z(x,t) \equiv \{\varepsilon^2 \delta_{x\overline{x}} - 1 - \delta_{\overline{t}}\} z(x,t) = 0, \; (x,t) \in G_h, \tag{1.12b}$$

$$z(x,t) = \varphi(x,t), \quad (x,t) \in S_h$$

is used. Here $\delta_{x\overline{x}}\, z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the second central and the first (backward) difference derivatives respectively, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$. The function $P(x,t)$ is approximated

$$P^{h+}(x,t) \equiv \varepsilon \delta_x z(x,t), \quad (x,t) \in \overline{G}_h^{\,-}, \tag{1.13}$$

where that function is defined, that is on $\overline{G}_h^{\,-} = \overline{\omega}_1^{\,-} \times \overline{\omega}_0$, where $\overline{\omega}_1^{\,-} = \overline{\omega}_1 \cap [0,d)$. In the case where $z(x,t)$ is the solution of (1.12), we use the notation $P^{h+}_{(1.13;1.12)}(x,t)$ or $P^{h+}_{(1.12)}(x,t)$ if this is not ambiguous.

Choosing T sufficiently large, we have the following inequality[23]:

$$|\; u_{(1.11)}(h,T) - z_{(1.12)}(h,T) \;| \geq m\,, \tag{1.14}$$

provided

$$\varepsilon = \varepsilon(h) = h\,, \tag{1.15}$$

and the inequality

$$|\; P_{(1.11)}(0,T) - P^{h+}_{(1.12)}(0,T) \;| \geq m \tag{1.16}$$

provided

$$\varepsilon = o\,(h) \;\text{ for }\; h \to 0\,. \tag{1.17}$$

It follows[23] that in the case when

$$T = T(\tau) = o\,(1) \;\text{ for }\; \tau \to 0, \tag{1.18}$$

the ratio of the real normalised diffusion flux on the boundary, namely $P(0,T)$, and the computed normalised flux $P^{h+}(0,T)$ increase unboundedly as $h, \tau \to 0$:

$$\frac{P_{(1.11)}(0,T)}{P^{h+}_{(1.12)}(0,T)} \to \infty \;\text{ for }\; h, \tau \to 0. \tag{1.19}$$

Thus, for differences of the functions $u_{(1.11)}(x,t) - z_{(1.12)}(x,t)$ and $P_{(1.11)}(x,t) - P^{h+}_{(1.12)}(x,t)$ and also for the ratios of the real flux and the computed flux the estimates (1.14), (1.16), (1.19) hold; that is the computed solution and flux do not converge ε-uniformly for $h,\ \tau \to 0$.

We summarise this in the following theorem.

Theorem 1.1 *The functions $z(x,t)$, $(x,t) \in \overline{G}_h$ and $P^{h+}(x,t)$, $(x,t) \in \overline{G}_h^{\,-}$ which are respectively the solution of the finite difference scheme (1.12) for (1.11), and the computed normalised diffusive flux, do not converge ε-uniformly to the functions $u(x,t)$ and $P(x,t)$, $(x,t) \in \overline{G}$, which are respectively the solution of boundary value problem (1.11) and the exact normalised diffusive flux. The ratio of the exact normalised flux and the computed flux is not bounded ε-uniformly when $h,\ \tau \to 0$.*

Remark 1. Instead of the function $P^{h+}(x,t)$, $(x,t) \in \overline{G}_h^{\,-}$, for the approximation of the flux $P(x,t)$ one can use the backward or central approximations

$$P^{h-}(x,t) = \varepsilon \delta_{\overline{x}} z(x,t), \quad (x,t) \in \overline{G}_h^{\,+}, \tag{1.20}$$

$$P^{h}(x,t) = \varepsilon \delta_{\check{x}} z(x,t), \quad (x,t) \in G_h, \tag{1.21}$$

where $\overline{G}_h^{\,+} = \overline{\omega}_1^{\,+} \times \overline{\omega}_0$, $\overline{\omega}_1^{\,+} = \omega_1 \cap (0,d]$,

$$\delta_{\check{x}} z(x,t) = \frac{z(x^{i+1},t) - z(x^{i-1},t)}{h^{i-1} + h^i}, \quad x = x^i \in \omega_1.$$

Also the functions $P^{h-}(x,t)$, $P^h(x,t)$ for $h,\ \tau \to 0$ do not converge to $P(x,t)$ ε-uniformly, for symmetry reasons.

Thus, in the case of the singularly perturbed boundary value problem (1.9) we arrive at the problem of developing special finite difference schemes which approximate ε-uniformly both the solution and the normalised diffusive flux.

2. Numerical Experiments with a Classical Difference Scheme

Firstly, let us formulate a problem suitable for numerical experiments with the classical finite difference schemes (1.11). The qualitative behaviour of the functions $z(x,t)$, $P^{h-}(x,t)$, $P^{h+}(x,t)$, $P^h(x,t)$ is described by Theorem 1.1 and Remark 1. It is interesting to analyze more precisely the errors of the approximate solution of (1.11) and the errors in the computed normalised flux. For the pointwise errors

$$e(x,t) = |u(x,t) - z(x,t)|, \quad (x,t) \in \overline{G}_h,$$

$$q^+(x,t) = |P(x,t) - P^{h+}(x,t)|, \quad (x,t) \in \overline{G}_h^{\,-}$$

the following inequalities hold

$$e(x,t) \le e^0(x) + |u(x,t) - u^0(x)| + |z(x,t) - z^0(x)|,$$

$$q^+(x,t) \le q^{0+}(x) + |P(x,t) - P^0(x)| + |P^{h+}(x,t) - P^{0\,h+}(x)|,$$

where

$$e^0(x) = e^0(x, \varepsilon, N) = |u^0(x) - z^0(x)|, \tag{2.22}$$

$$q^{0\,+}(x) = q^{0\,+}(x, \varepsilon, N) = |P^0(x) - P^{0\,h+}(x)|. \tag{2.23}$$

Here, the function $u^0(x)$ is the solution of the stationary problem

$$\begin{aligned} L_{(2.24)}u(x) \quad &\equiv \{\, \varepsilon^2 \tfrac{d^2}{dx^2} - 1 \,\} u(x) = 0, \quad x \in D, \\ u(x) \quad &= \varphi(x), \qquad\qquad\qquad x \in \Gamma, \end{aligned} \tag{2.24}$$

$\Gamma = \overline{D} \setminus D$, the function $P^0(x)$ is the normalised diffusive flux for stationary problem (2.24): $P^0(x) = \varepsilon (d/dx) u^0(x)$, $x \in \overline{D}$. The boundary function $\varphi(x)$ in problem (2.24) is defined as in (1.11).

The function $z^0(x)$ is the solution of the stationary discrete problem

$$\begin{aligned} \Lambda_{(2.25)} z(x) \quad &\equiv \{\, \varepsilon^2 \delta_{x\overline{x}} - 1 \,\} z(x) = 0, \\ &x \in D_h, \\ z(x) \quad &= \varphi(x), \qquad\qquad x \in \Gamma_h, \end{aligned} \tag{2.25}$$

the function $P^{0\,h+}(x)$ is the normalised diffusive flux for problem (2.25): $P^{0\,h+}(x) = \varepsilon \delta_x z^0(x)$, $x \in \overline{D}_h^{\,-}$. The largest contribution to the functions $e(x,t)$ and $q^+(x,t)$ for $t = T$, with T is sufficiently large, and small h and τ, is caused by the terms $e^0(x)$ and $q^{0\,+}(x)$. Therefore the main interest here is in the numerical investigation of the influence of the parameter ε and the number N on the behaviour of values $e^0(x; \varepsilon, N)$ and $q^{0+}(x; \varepsilon, N)$. As the derivatives of the function $u^0(x)$ become large only for small values of ε, it is most interesting to investigate the behaviour of $e^0(x, \varepsilon, N)$, $q^{0+}(x, \varepsilon, N)$ for this case.

The behaviour of the general errors $e^0(x, \varepsilon, N)$, $q^{0+}(x, \varepsilon, N)$ is complex and not particularly suitable for direct analysis of the numerical results. Therefore instead of problem (2.24), (1.11) we consider a closely related problem for which analysis of the errors for the approximate solutions and fluxes is considerably simpler. Let the function

$$W(x) = \exp(-\varepsilon^{-1} x), \quad x \in \overline{D}$$

be the solution of singularly perturbed equation (2.24). Then

$$\max_{\overline{D}} |W(x)| = W(0) = 1, \ \max_{\overline{D}} |\varepsilon \frac{d}{dx} W(x)| = -\varepsilon \frac{d}{dx} W(0) = 1. \tag{2.26}$$

Further, the following estimate holds for $u^0(x)$,

$$|u^0(x) - W(x)|, \ |\varepsilon \frac{d}{dx} u^0(x) - \varepsilon \frac{d}{dx} W(x)| \le M\, \varepsilon^n, \quad x \in \overline{D}, \tag{2.27}$$

where n is a sufficiently large number. Thus, the function $W(x)$ and $\varepsilon(d/dx)W(x)$ approximate well the solution of problem (2.24) and the normalised diffusive flux $P^0(x)$, for sufficiently small ε.

By virtue of monotonicity of the operator $\Lambda_{(2.25)}$ for the solution of the difference scheme

$$\begin{aligned} \Lambda_{(2.25)} z(x) \quad &= \quad 0, \qquad\quad x \in D_h, \\ z(x) \quad &= \quad W(x), \quad x \in \Gamma_h \end{aligned} \tag{2.28}$$

the following estimates

$$
\begin{aligned}
&|z_{(2.28)}(x) - z_{(2.25)}(x)| \le M\varepsilon^n, \quad x \in \overline{D}_h, \\
&|\, P^{h+}_{(2.28)}(0) - P^{h+}_{(2.25)}(0) \,| \le M\varepsilon^n,
\end{aligned}
\tag{2.29}
$$

are valid where $P^{h+}_{(2.28)}(0)$ and $P^{h+}_{(2.25)}(0)$ are the normalised diffusive fluxes for problems (2.28) and (2.25). According to relations (2.27), (2.29), the principal parts of the errors $|u_{(1.11)}(x,T) - z_{(1.12)}(x,T)|$ and $|P(0,T) - P^{h+}_{(1.12)}(0,T)|$ for sufficiently large T and small values of ε, are the errors $|W(x) - z_{(2.28)}(x)|$ and $|\varepsilon(d/dx)W(0) - P^{h+}_{(2.28)}(0)|$.

Therefore, let us consider the difference scheme (2.28) for the boundary value problem (2.24) with boundary condition

$$
\varphi(x) = W(x), \quad x \in \Gamma. \tag{2.30}
$$

We wish to demonstrate the influence of the parameter ε and the number of nodes N on the error of the approximate solution and also on the error of computed normalised flux at $x = 0$.

Suppose

$$
\begin{aligned}
e(x) &= e(x;\varepsilon,N) = |u(x) - z(x)|, \\
q^+(x) &= q^+(x;\varepsilon,N) = |P(x) - P^{h+}(x)|,
\end{aligned}
$$

where $u(x)$ is the solution of problem (2.24), (2.30), $z(x) = z_{(2.28)}(x)$ is the solution of difference scheme (2.28), and $P(x)$ and $P^{h+}(x) = P^{h+}_{(2.28)}(x)$ are the normalised fluxes for problems (2.24), (2.30) and (2.28). Note that $u(x) = W(x)$, $x \in \overline{D}$. Using (2.26) it is clear that the solution of problem (2.24), (2.30) satisfies the following conditions

$$
\max_{\overline{D}} |u(x)| = u(0) = 1, \quad \max_{\overline{D}} |P(x)| = -P(0) = 1.
$$

In Tables 4 and 5 we can see the results of computing the errors $E(\varepsilon, N)$

$$
E(\varepsilon,N) = \max_{\overline{D}_h} e(x;\varepsilon,N) \tag{2.31a}
$$

that are the maximal pointwise errors on the grid $\overline{D}_h$, and results of computing the errors $Q(\varepsilon, N)$

$$
Q(\varepsilon,N) = q^+(x=0;\varepsilon,N), \tag{2.32a}
$$

that is the errors of the normalised flux on the boundary $x = 0$. These results were obtained using the difference scheme (2.28) for various values of ε and N. The values of $\overline{E}(N)$ and $\overline{Q}(N)$ are also given, where

$$
\overline{E}(N) = \max_{\varepsilon=4^{-m}} E(\varepsilon,N), \quad m = 0,1,\cdots,6, \tag{2.31b}
$$

is the largest (with respect to ε) error of the approximate solution (for a fixed value of N, and

$$
\overline{Q}(N) = \max_{\varepsilon=4^{-m}} Q(\varepsilon,N), \quad m = 0,1,\cdots,6, \tag{2.32b}
$$

Table 4: Table of errors $E(\varepsilon, N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	3.621e−04	2.296e−05	1.437e−06	8.982e−08	5.614e−09
2^{-2}	1.395e−02	9.443e−04	5.934e−05	3.710e−06	2.319e−07
2^{-4}	3.741e−02	1.409e−02	9.526e−04	5.985e−05	3.742e−06
2^{-6}	3.876e−03	3.741e−02	1.409e−02	9.526e−04	5.985e−05
2^{-8}	2.440e−04	3.876e−03	3.741e−02	1.409e−02	9.526e−04
2^{-10}	1.526e−05	2.440e−04	3.876e−03	3.741e−02	1.409e−02
2^{-12}	9.537e−07	1.526e−05	2.440e−04	3.876e−03	3.741e−02
$\overline{E}(N)$	3.741e−02	3.741e−02	3.741e−02	3.741e−02	3.741e−02

Table 5: Table of errors of the normalised flux $Q(\varepsilon, N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.164e−01	3.071e−02	7.779e−03	1.951e−03	4.881e−04
2^{-2}	3.818e−01	1.172e−01	3.076e−02	7.782e−03	1.951e−03
2^{-4}	7.639e−01	3.820e−01	1.172e−01	3.076e−02	7.782e−03
2^{-6}	9.377e−01	7.639e−01	3.820e−01	1.172e−01	3.076e−02
2^{-8}	9.844e−01	9.377e−01	7.639e−01	3.820e−01	1.172e−01
2^{-10}	9.961e−01	9.844e−01	9.377e−01	7.639e−01	3.820e−01
2^{-12}	9.990e−01	9.961e−01	9.844e−01	9.377e−01	7.639e−01
$\overline{Q}(N)$	9.990e−01	9.961e−01	9.844e−01	9.377e−01	7.639e−01

is the largest error of the computed normalised flux for $x = 0$. The values $\overline{E}(N)$ and $\overline{Q}(N)$ define the best guaranteed accuracy which can be obtained using the scheme (2.28) to solve the problem (2.24), (2.30) for a given N and all values of ε shown.

From Table 4 we see that the solution of difference scheme (2.28) converges to the solution of boundary value problem (2.24), (2.30) for a fixed value of the parameter ε. However, these approximate solutions $z(x) = z(x; \varepsilon, N)$ do not converge ε-uniformly. For the worst realisable error $\overline{E}(N)$ the lower bound $\overline{E}(N) \geq 3.74 \cdot 10^{-2}$ holds. The relative worst realisable error for a fixed N is given by formula

$$\delta(N) \equiv \frac{\overline{E}(N)}{\max_{\overline{D}} |u(x)|},$$

where $u(x)$ is the solution of problem (2.24), (2.30). The relative error $\delta(N)$ does not depend on N and it is equal to 3.741 %.

From Table 5 it follows that $P^{h+}(0) = P^{h+}(0; \varepsilon, N)$, the computed normalised diffusive flux at the boundary $x = 0$, converges to the value $P(0)$, with increasing N for fixed ε. However, these computed fluxes $P^{h+}(x) = P^{h+}(x; \varepsilon, N)$ also do not converge ε-uniformly. The error $Q(\varepsilon, N)$ is constant for any value of N if the product εN is constant. Moreover, for any fixed N the error $Q(\varepsilon, N)$ tends to the value $\mid P(0) \mid = 1$ as ε increases.

Table 6 gives the values of

$$\lambda(\varepsilon, N) \equiv \frac{|P(0;\varepsilon)|}{|P^{h+}(0;\varepsilon,N)|} = \frac{|\,\frac{d}{dx}u(0;\varepsilon)\,|}{|\delta_x z(0;\varepsilon,N)|}$$

which is the ratio of the exact normalised flux on the boundary $x = 0$ to the computed flux (or the ratio of the first derivative of the exact solution at $x = 0$ to the computed first difference).

Table 6: Table of ratios normalised fluxes $\lambda(\varepsilon, N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.132e+00	1.032e+00	1.008e+00	1.002e+00	1.000e+00
2^{-2}	1.618e+00	1.133e+00	1.032e+00	1.008e+00	1.002e+00
2^{-4}	4.236e+00	1.618e+00	1.133e+00	1.032e+00	1.008e+00
2^{-6}	1.606e+01	4.236e+00	1.618e+00	1.133e+00	1.032e+00
2^{-8}	6.402e+01	1.606e+01	4.236e+00	1.618e+00	1.133e+00
2^{-10}	2.560e+02	6.402e+01	1.606e+01	4.236e+00	1.618e+00
2^{-12}	1.024e+03	2.560e+02	6.402e+01	1.606e+01	4.236e+00
$\overline{\lambda}(N)$	1.024e+03	2.560e+02	6.402e+01	1.606e+01	4.236e+00

From Table 6 we see that the value $\lambda(\varepsilon, N)$ satisfies the relation $\lambda(\varepsilon, N) \approx \varepsilon^{-1} N^{-1}$ and increases unboundedly for any fixed N and sufficiently small values of ε satisfying $\varepsilon N \to 0$. Even for $\varepsilon \leq 16^{-1}h$ the real flux differs from the computed flux by a factor of 10.

The relative error of the flux $\eta(\varepsilon, N)$, where

$$\eta(\varepsilon, N) \equiv \frac{Q(\varepsilon, N)}{|P(0)|} = Q(\varepsilon, N),$$

can be guaranteed to be no larger than 20 % only for $N \geq N(\varepsilon) = 4\varepsilon^{-1}$.

Thus, the results presented illustrate the statements of Theorem 1.1 and demonstrate the weaknesses of classical difference schemes for the solution of problems of the form (1.11). Since $|P(0)| = 1$, the results of table 5 show that if we use the classical difference scheme (1.12) for solving problem (1.11), in the case where $\alpha_0 \leq \varepsilon \leq \alpha_1$, $\alpha_0 = 2.4 \cdot 10^{-4}$, $\alpha_1 = 1.56 \cdot 10^{-2}$ (which corresponds to the diffusion Fourier number $F_O^D = 5.76 \cdot 10^{-8} - 2.43 \cdot 10^{-4}$ for the phenol diffusion process discussed above), we cannot guarantee an error in the computed normalised flux through the boundary that is less than 50%, even when the number of nodes is $N = 1024$.

3. Grid Approximations of Solutions and Diffusive Fluxes

In this section, we construct special finite difference schemes for problems (1.9), (1.10) and computational formulae for the approximation of the normalised diffusion flux. We suppose that $u \in C^{4,2}(\overline{G})$ for each fixed value of ε, $\varepsilon \in (0, 1]$.

On the set $\overline{G}$ we introduce again the grid

$$\overline{G}_h = \overline{\omega}_1 \times \overline{\omega}_0, \tag{3.33}$$

where now $\overline{\omega}_1$ is a grid, generally nonuniform, on the interval $[0, d]$ and $\overline{\omega}_0$ is an uniform grid on the interval $[0, T]$. Suppose $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}_1$, $h = \max_i h^i$. By $N+1$ and N_0+1 we denote the number of nodes in the grids $\overline{\omega}_1$ and $\overline{\omega}_0$ respectively, $h \le M N^{-1}$.

On the grid $\overline{G}_h$ we define the following difference scheme for problem (1.9),

$$\begin{aligned} \Lambda_{(3.34)} z(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{3.34}$$

Here $G_h = G \cap \overline{G}_h, \quad S_h = S \cap \overline{G}_h,$

$$\Lambda_{(3.34)} z(x,t) \equiv \varepsilon^2 a(x,t) \delta_{\overline{x}\widehat{x}} z(x,t) - c(x,t) z(x,t) - p(x,t) \delta_{\overline{t}} z(x,t).$$

To approximate the function $P(x,t)$, that is the normalised diffusive flux, we use the grid function $P^{h+}_{(1.13;3.34)}(x,t)$.

The difference scheme (3.34), (3.33) is monotonic for any arbitrary distribution of the nodes of the grid $\overline{\omega}_1$ and hence of the grid $\overline{G}_{h\,(3.33)}$. Using the maximum principle we establish convergence of the difference scheme for a fixed value of the parameter

$$|u(x,t) - z(x,t)| \le M\,[\, \varepsilon^{-1} N^{-1} + N_0^{-1}\,], \ (x,t) \in \overline{G}_h. \tag{3.35}$$

In the case of the grid

$$\overline{G}_h = \{\, G_{h\,(3.33)}, \text{ where } \overline{\omega}_1 \text{ is an uniform grid}\} \tag{3.36}$$

the estimate

$$|u(x,t) - z(x,t)| \le M\,[\, \varepsilon^{-2} N^{-2} + N_0^{-1}\,], \ (x,t) \in \overline{G}_{h\,(3.36)} \tag{3.37}$$

holds. From (3.37), the inequality

$$|P(x,t) - P^{h+}(x,t)| \le M\,[\, \varepsilon^{-1} N^{-1} + \varepsilon N N_0^{-1}\,], \ (x,t) \in \overline{G}^{-}_{h\,(3.36)} \tag{3.38}$$

follows. A sufficient condition for convergence of the function $P^{h+}(x,t)$ to the function $P(x,t)$ for a fixed value of ε, is that

$$N N_0^{-1} \to 0 \text{ for } N, N_0 \to \infty. \tag{3.39}$$

Thus, the difference scheme (3.34), (3.36), (3.39) allows approximation of the solution of boundary value problem (1.9) together with the normalised diffusive flux for a fixed value of ε. In particular, under the condition

$$N_0 = N_0(N) = N^2 \tag{3.40}$$

the following estimate

$$|P(x,t) - P^{h+}(x,t)| \le \frac{M}{\varepsilon N}, \quad (x,t) \in \overline{G}^{-}_{h\,(3.36)}. \tag{3.41}$$

is valid.

In the case of boundary value problem (1.10), we use the difference scheme

$$\begin{aligned} \Lambda_{(3.42)}(z(x,t)) &= 0, \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{3.42}$$

Here

$$\Lambda_{(3.42)}(z(x,t)) \equiv \{ \varepsilon^2 a(x,t)\delta_{\overline{x}\widehat{x}} - p(x,t)\delta_{\overline{t}} \} z(x,t) - g(x,t,z(x,t)).$$

For the solution of (3.42) and for the flux $P(x,t)$ the bounds (3.35), (3.37), (3.38) also hold. When the condition (3.39) is violated then the function $P^{h+}(x,t)$, $(x,t) \in \overline{G}_h^{\,-}$ does not, in general, converge to the function $P(x,t)$ for a fixed ε. The main result is summarised in the following theorem.

Theorem 3.1 *Let the finite difference scheme (3.34), (3.33) (or (3.42), (3.33)) be used for the solution of the boundary value problem (1.9) (respectively (1.10)). Then condition (3.39) is sufficient for convergence of $P^{h+}(x,t)$ for a fixed value of the parameter, if $u \in C^{4,2}(\overline{G})$ and the grid (3.36) is used. Moreover, estimate (3.41) holds if (3.40) is satisfied.*

Now we construct a special difference scheme for problem (1.9). On the grid $\overline{G}_h$ we introduce a special grid, condensed in the boundary layer, similar to the grid constructed in [21, 22],

$$\overline{G}_{h\,(3.43)} = \overline{G}_{h\,(3.43)}(\sigma) = \overline{\omega}_1^* \times \overline{\omega}_0, \tag{3.43a}$$

where $\overline{\omega}_1^* = \overline{\omega}_1^*(\sigma)$ is a piecewise uniform grid on $[0,d]$; σ is a parameter depending on ε and N. Step-sizes of the grid $\overline{\omega}_1^*$ on the intervals $[\ 0,\sigma\ ]$, $[\ d-\sigma, d\ ]$ and on the interval $[\ \sigma, d-\sigma\ ]$ are constant and equal to $h^{(1)} = 4\sigma N^{-1}$ and $h^{(2)} = 2(d-2\sigma)N^{-1}$ respectively, $\sigma \le 4^{-1}d$. The value σ is chosen to satisfy the condition

$$\sigma = \sigma_{(3.43)}(\varepsilon, N) = \min[\ 4^{-1}d,\ m^{-1}\varepsilon \ln N\ ], \tag{3.43b}$$

where $m = m_{(3.43)}$ is an arbitrary number.

In a manner similar to that in [23]we establish the ε-uniform convergence of the scheme (3.34), (3.43)

$$|u(x,t) - z(x,t)| \le M[\ N^{-2}\ln^2 N + N_0^{-1}\ ],\ (x,t) \in \overline{G}_h. \tag{3.44}$$

For the computed flux, we have

$$|P(x,t) - P^{h+}(x,t)| \le M\varepsilon[\ N^{-1}\ln^2 N + N\,N_0^{-1}\ ],\ (x,t) \in \overline{G}_{h(3.43)}^{\,-}. \tag{3.45}$$

According to the estimate (3.45), we have ε-uniform convergence of the function $P^{h+}(x,t)$, provided the condition

$$\varepsilon N N_0^{-1} \to 0 \quad \varepsilon\text{-uniformly, for } N, N_0 \to \infty \tag{3.46}$$

is fulfilled. In particular, under condition (3.40) the estimate

$$|P(x,t) - P^{h+}(x,t)| \le M\,\varepsilon\,N^{-1}\ln^2 N \le M\,N^{-1}\ln^2 N,\quad (x,t) \in \overline{G}_{h(3.43)}^{\,-}. \tag{3.47}$$

is valid.

Note that estimates (3.44), (3.45), (3.47) are also fulfilled in the case of the boundary value problem (1.10), when the scheme (3.42), (3.43) is used for solving of this problem. Thus we have the following theorem [23].

Theorem 3.2 Let $u \in C^{4,2}(\overline{G})$ for a fixed value of the parameter ε, $\varepsilon \in (0,1]$. Then the solution of difference scheme (3.34), (3.43) (or (3.42), (3.43)) converges ε-uniformly to the solution of the problem (1.9) (respectively (1.10). If condition (3.46) also holds, $P^{h+}(x,t)$, $(x,t) \in \overline{G}^{-}_{h(3.43)}$, converges ε-uniformly to the function $P(x,t)$. For the solution of the difference scheme the estimates (3.37), (3.44) and, for computed flux $P^{h+}(x,t)$, the estimates (3.41) (if condition (3.40) is fulfilled) and (3.47) are valid . For the flux the estimates (3.38), (3.45) also hold. Estimates similar to (3.38), (3.45) and (3.41), (3.47) also hold if $P^{h-}_{(1.20)}(x,t)$ or $P^{h}_{(1.21)}(x,t)$ are used as approximations of $P(x,t)$.

It is also interesting investigate numerically the influence of ε and N on the behaviour of $E_{(2.31)}(\varepsilon,N)$ and $Q_{(2.32)}(\varepsilon,N)$ computed using the special difference scheme for problem (2.24), (2.30). We use the difference scheme

$$\begin{aligned} \Lambda_{(3.48)}z(x) &\equiv \{\varepsilon^2\delta_{\overline{x}\widehat{x}} - 1\}z(x) = 0, \quad x \in D_h, \\ z(x) &= W(x), \qquad\qquad\qquad x \in \Gamma_h, \end{aligned} \tag{3.48a}$$

on the grid

$$\overline{D}_h = \overline{\omega}^*_{1(3.43)}, \tag{3.48b}$$

where $m_{(3.43)} = 1/2$. In the Tables 7, 8 and 9 we give the values $E(\varepsilon,N)$, $Q(\varepsilon,N)$, $\overline{E}(N)$, $\overline{Q}(N)$, $\lambda(\varepsilon,N)$, computed with (3.48) for various values of ε and N.

Table 7: Table of errors $E(N,\varepsilon)$ for the special scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	3.621e−04	2.296e−05	1.437e−06	8.982e−08	5.614e− 09
2^{-2}	1.395e−02	9.443e−04	5.934e−05	3.710e−06	2.319e− 07
2^{-4}	1.685e−02	1.409e−02	9.526e−04	5.985e−05	3.742e− 06
2^{-6}	4.050e−02	2.410e−02	4.041e−03	4.587e−04	4.492e− 05
2^{-8}	5.690e−02	2.406e−02	4.041e−03	4.587e−04	4.492e− 05
2^{-10}	6.109e−02	2.404e−02	4.041e−03	4.587e−04	4.492e− 05
2^{-12}	6.215e−02	2.404e−02	4.041e−03	4.587e−04	4.492e− 05
$\overline{E}(N)$	6.215e−02	2.410e−02	4.041e−03	4.587e−04	4.492e−05

From the Tables 7, 8 we can see that approximate solutions and computed normalised fluxes seem to converge ε-uniformly. For example, the guaranteed accuracy for the approximate solution is not worse than 1.0 % ,when $N = 64$, and for the computed

Table 8: Table of errors of the normalised flux $Q(N,\varepsilon)$ for the special sch eme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.164e−01	3.071e−02	7.779e−03	1.951e−03	4.881e− 04
2^{-2}	3.818e−01	1.172e−01	3.076e−02	7.782e−03	1.951e− 03
2^{-4}	6.679e−01	3.820e−01	1.172e−01	3.076e−02	7.782e− 03
2^{-6}	6.473e−01	4.764e−01	2.267e−01	8.290e−02	2.671e− 02
2^{-8}	6.413e−01	4.763e−01	2.267e−01	8.290e−02	2.671e− 02
2^{-10}	6.398e−01	4.763e−01	2.267e−01	8.290e−02	2.671e− 02
2^{-12}	6.395e−01	4.763e−01	2.267e−01	8.290e−02	2.671e− 02
$\overline{Q}(N)$	6.679e−01	4.764e−01	2.267e−01	8.290e−02	2.671e−02

Table 9: Table of ratios of the normalised fluxes $\lambda(N,\varepsilon)$ for the spe cial scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.132e+00	1.032e+00	1.008e+00	1.002e+00	1.000e+00
2^{-2}	1.618e+00	1.133e+00	1.032e+00	1.008e+00	1.002e+00
2^{-4}	3.012e+00	1.618e+00	1.133e+00	1.032e+00	1.008e+00
2^{-6}	2.835e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00
2^{-8}	2.788e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00
2^{-10}	2.776e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00
2^{-12}	2.774e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00
$\overline{\lambda}(N)$	3.012e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00

flux is not worse than 10% for $N = 256$. From Table 9 we see that $\lambda(\varepsilon, N)$ tends ε-uniformly to 1 with increasing N.

4. A Numerical Example for the Diffusion Equation

In order to illustrate computational problems which appear with employment of classical difference schemes to solve singularly perturbed boundary value problems for a partial differential equation and to find normalised fluxes, and in order to show the efficiency of special difference schemes we shall consider the simplest boundary value problem for the diffusion equation. The function

$$W(x,t) = \operatorname{erfc}(\frac{x}{2\varepsilon\sqrt{t}})(\frac{x^2}{2\varepsilon^2}+t) - \frac{1}{\sqrt{\pi}}\exp(-\frac{x^2}{4\varepsilon^2 t})\frac{x\sqrt{t}}{\varepsilon}, \quad 0 < x < \infty, \ t \geq 0$$

is the solution of the singularly perturbed diffusion equation

$$L_{(4.49)}u(x,t) \equiv \varepsilon^2\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial t}u(x,t) = 0, \quad 0 < x < \infty, \ t > 0 \tag{4.49}$$

and satisfies the boundary conditions

$$W(x,0) = 0, \ 0 \leq x < \infty, \quad W(0,t) = t, \ t \geq 0.$$

For the function $W(x,t)$ the following bounds hold

$$\max_{\substack{0<x<\infty,\\ 0\le\vartheta\le t}} W(x,\vartheta) \le t,$$

$$\max_{0<x<\infty} \varepsilon \left| \frac{\partial}{\partial x} W(x,t) \right| \le \varepsilon \left| \frac{\partial}{\partial x} W(0,t) \right| = 2\,\pi^{-1/2}\,t^{1/2}, \quad t \ge 0.$$

For $x \ge x_0 > m,\ 0 \le t \le T$, the function $W(x,t)$ decays more rapidly than any power of the parameter ε that is

$$|\, W(x,t)\,| \le M\,\varepsilon^n, \quad x \ge x_0 > m, \quad 0 \le t \le T,$$

where n is an arbitrary large number.

Let us consider the boundary value problem

$$\begin{aligned} L_{(4.49)} u(x,t) &= 0, & (x,t) &\in G, \\ u(x,t) &= W(x,t), & (x,t) &\in S, \end{aligned} \tag{4.50}$$

where $G = D \times (\,0, T\,],\ D = (0,d),\ d = 1,\ T = 1$.

The difference scheme (3.34), (3.33) for problem (4.50) is

$$\begin{aligned} \Lambda_{(4.51)} z(x,t) \equiv \varepsilon^2 \delta_{\overline{x}\widehat{x}} z(x,t) - \delta_{\overline{t}} z(x,t) &= 0, \quad (x,t) \in G_h, \\ z(x,t) = W(x,t), &\qquad\ (x,t) \in S_h. \end{aligned} \tag{4.51}$$

Here $\overline{G}_h$ is one of the grids considered previously, either the uniform grid $\overline{G}_{h\,(3.36)}$ or the special grid $\overline{G}_{h\,(3.43)} = \overline{G}_{h\,(3.43)}$ with $m_{(3.43)} = 1/2$. Using the solutions of the difference schemes on these meshes, we calculated the values

$$E(\varepsilon, N) = \max_{\overline{G}_h} |u(x,t) - z(x,t)|,$$

which are the errors of the approximate solution (l^∞-norm) for various values of ε and $N = N_0$, and also the values

$$Q(\varepsilon, N) = \max_{0\le t\le T} |\, P(x=0,t) - P^{h+}(x=0,t)\,|,$$

which are the errors in the computed normalised flux on the boundary $x = 0$, where $P(x,t) = \varepsilon(\partial/\partial x)u(x,t),\ P^{h+}(x,t) = \varepsilon\delta_x z(x,t)$.

In the Tables 10 and 11 we show the values of $E(\varepsilon, N)$ and $Q(\varepsilon, N)$ computed with the uniform grid $\overline{G}_{h\,(3.36)}$ for various values of ε and $N = N_0$. In the Tables 12, 13 results for the special grid $\overline{G}_{h\,(3.43)}$ are given.

From tables 10, 11 we can see that the solution of the difference scheme (4.51), (3.36) for $N = N_0$ and also the computed normalised flux for $x = 0$ converge for a fixed value of the parameter. However, approximate solutions and normalised fluxes do not converge ε-uniformly. For the $\overline{E}(N) = \max_{\varepsilon=4^{-m}} E(\varepsilon, N),\ m = 0, 1, \ldots, 6$, we find

$$\overline{E}(N) \ge 2.9 \cdot 10^{-2}.$$

Table 10: Table of errors $E(\varepsilon, N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1.0	1.630e−02	6.144e−03	1.780e−03	4.651e−04	1.176e−04
2^{-2}	4.374e−02	8.624e−03	1.960e−03	4.769e−04	1.184e−04
2^{-4}	3.601e−02	2.558e−02	3.131e−03	5.507e−04	2.484e−04
2^{-6}	2.432e−03	3.095e−02	2.061e−02	1.728e−03	2.444e−04
2^{-8}	1.526e−04	2.069e−03	2.966e−02	1.934e−02	1.376e−03
2^{-10}	9.537e−06	1.297e−04	1.978e−03	2.934e−02	1.902e−02
2^{-12}	5.960e−07	8.106e−06	1.240e−04	1.956e−03	2.926e−02
$\overline{E}(N)$	4.374e−02	3.095e−02	2.966e−02	2.934e−02	2.926e−02

Table 11: Table of errors of the normalised flux $Q(\varepsilon, N)$, $\overline{Q}(N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1.0	1.620e−01	6.123e−02	2.362e−02	9.946e−03	4.496e−03
2^{-2}	4.516e−01	1.328e−01	4.345e−02	1.535e−02	5.904e−03
2^{-4}	8.876e−01	4.332e−01	1.228e−01	3.321e−02	1.086e− 02
2^{-6}	1.066e+00	8.863e−01	4.282e−01	1.211e−01	3.111e−02
2^{-8}	1.113e+00	1.066e+00	8.860e−01	4.270e−01	1.207e−01
2^{-10}	1.124e+00	1.113e+00	1.066e+00	8.859e−01	4.267e−01
2^{-12}	1.127e+00	1.124e+00	1.113e+00	1.066e+00	8.859e−01
$\overline{Q}(N)$	1.127e+00	1.124e+00	1.113e+00	1.066e+00	8.859e−01

The ratio of the exact normalised flux on the boundary $x = 0$ for $t = T$ and the computed flux (that is $P(0,T)/P^{h+}(0,T)$) increases unboundedly with decreasing ε, for fixed values of N.

Table 12: Table of errors $E(\varepsilon, N)$ for the special scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.630e−02	6.144e−03	1.780e−03	4.651e−04	1.176e−04
2^{-2}	4.374e−02	8.624e−03	1.960e−03	4.769e−04	1.184e−04
2^{-4}	3.976e−02	2.558e−02	3.131e−03	5.507e−04	2.484e−04
2^{-6}	4.494e−04	4.156e−02	7.214e−03	1.077e−03	2.478e−04
2^{-8}	9.440e−03	4.156e−02	7.214e−03	1.077e−03	2.478e−04
2^{-10}	1.207e−02	4.156e−02	7.214e−03	1.077e−03	2.478e−04
2^{-12}	1.273e−02	4.156e−02	7.214e−03	1.077e−03	2.478e−04
$\overline{E}(N)$	4.374e−02	4.156e−02	7.214e−03	1.077e−03	2.478e−04

From the Tables 12 and 13 we see that approximate solutions and computed normalised fluxes (for $x = 0$) seem to converge ε-uniformly.

Table 13: Table of errors of the normalised flux $Q(\varepsilon, N)$, $\overline{Q}(N)$ for the special scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.620e−01	6.123e−02	2.362e−02	9.946e−03	4.496e−03
2^{-2}	4.516e−01	1.328e−01	4.345e−02	1.535e−02	5.904e−03
2^{-4}	7.867e−01	4.332e−01	1.228e−01	3.321e−02	1.086e−02
2^{-6}	7.727e−01	5.505e−01	2.428e−01	8.507e−02	2.701e−02
2^{-8}	7.690e−01	5.505e−01	2.428e−01	8.507e−02	2.701e−02
2^{-10}	7.680e−01	5.505e−01	2.428e−01	8.507e−02	2.701e−02
2^{-12}	7.678e−01	5.505e−01	2.428e−01	8.507e−02	2.701e−02
$\overline{Q}(N)$	7.867e−01	5.505e−01	2.428e−01	8.507e−02	2.701e−02

5. Boundary Value Problem on a Rectangle

In this section we consider a quasi-linear parabolic equations on a rectangle. We shall point out computational problems accompanying the flux computation, and shall construct special difference schemes whose approximate solutions and computed normalised fluxes converge ε-uniformly.

On the rectangle

$$D = \{\ x :\ 0 < x_s < d_s,\ s = 1, 2\ \}$$

we consider the boundary value problem for the quasi-linear equation of parabolic type

$$\begin{aligned} L_{(5.52)}(u(x,t)) &= 0, \quad (x,t) \in G, \\ u(x,t) &= \varphi(x,t), \quad (x,t) \in S. \end{aligned} \tag{5.52a}$$

Here

$$G = D \times (\ 0, T\ ], \quad S = \overline{G} \setminus G, \tag{5.53}$$

$$L_{(5.52)}(u(x,t)) \equiv \{\ \varepsilon^2 L^2_{(5.52)} - p(x,t)\frac{\partial}{\partial t}\ \} u(x,t) - g(x,t,u(x,t)),$$

$$L^2_{(5.52)} \equiv \sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2} + \sum_{s=1,2} b_s(x,t)\frac{\partial}{\partial x_s} - c^0(x,t),$$

the functions $a_s(x,t)$, $b_s(x,t)$, $c^0(x,t)$, $p(x,t)$, $s = 1, 2$, and also the functions $g(x,t,u)$, $\varphi(x,t)$ are sufficiently smooth functions on the sets $\overline{G}$, $\overline{G} \times I\!R$ and S respectively. In addition, we shall assume that

$$a_0 \le a_1(x,t),\ a_2(x,t) \le a^0,\ c^0(x,t) \ge 0,\ p(x,t) \ge p_0,\ \ (x,t) \in \overline{G},\ a_0,\ p_0 > 0,$$

$$-M_{(5.52)} \le \frac{\partial}{\partial u} g(x,t,u) < \infty, \quad (x,t,u) \in \overline{G} \times R. \tag{5.52b}$$

The parameter ε takes arbitrary values in the interval (0,1]. Let $\Gamma = \overline{D} \setminus D$ and Γ^* be the set of corner points of the rectangle D, then assume that on the set $S^* = S_0^* \cup S_1^*$, $S_0^* = \{(x,t) : \ x \in \Gamma, \ t = 0\}$, $S_1^* = \{(x,t) : \ x \in \Gamma^*, \ 0 < t \le T\}$, the usual compatibility conditions are satisfied so that smoothness of the solution is ensured for each fixed value of the ε.

The solution of the boundary value problem is the function $u(x,t)$, $(x,t) \in \overline{G}$ such that $u \in C^{2,1}(G) \cap C^{1,0}(\overline{G})$, and also this function is assumed to satisfy an equation on G at $t = 0$, and a boundary condition on S. As ε tends to zero, a parabolic boundary layer appears in the neighbourhood of the set S_1

It is required to find the solution of the boundary value problem and also its normalised gradient

$$P_s(x,t) = \varepsilon \frac{\partial}{\partial x_s} u(x,t), \quad (x,t) \in \overline{G}, \quad s = 1,2.$$

One-sided differences are used for approximation of the first order spatial derivatives. For these schemes the accuracy of the approximate solution is normally not greater than first order. In the presence of corner points (or edges) the solution smoothness is reduced, thus causing a decrease in the convergence order for the numerical methods. It can be found that second order spatial derivatives are bounded in a neighbourhood of corner points (for a fixed value of the parameter), however in this case also the order of convergence is not greater than one with respect to the spatial variables.

For difference schemes for which the convergence order is no higher than one (with respect to the spatial variables), the first order difference derivatives do not necessarily converge with increasing the number of grid nodes. Therefore the difference derivatives of the computed solution cannot be used for the approximation of fluxes. Thus, in this case, the issue of constructing acceptable difference approximations of the diffusive fluxes appears.

On the set $\overline{G}_{(5.53)}$, we introduce the grid

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}_1 \times \overline{\omega}_2 \times \overline{\omega}_0, \tag{5.54}$$

where $\overline{\omega}_s$ is a grid, in general nonuniform, on the interval $[\,0, d_s\,]$ on axis x_s, $s = 1,2$, and $\overline{\omega}_0$ is an uniform grid on the interval $[\,0, T\,]$ on axis t with a step-size $\tau = TN_0^{-1}$. We denote by $h_s^i = x_s^{i+1} - x_s^i$, x_s^i, $x_s^{i+1} \in \overline{\omega}_s$, $h_s = \max_i h_s^i$, $h = \max h_s$, $s = 1,2$. By $N_s + 1$ we denote the number of nodes of the grid $\overline{\omega}_s$; $N = \min_s N_s$, $s = 1,2$, $h \le MN^{-1}$. For problem (5.52) we consider the difference scheme on the grid $\overline{G}_h$ given by

$$\begin{aligned} \Lambda_{(5.55)}(z(x,t)) &= 0, & (x,t) &\in G_h, \\ z(x,t) &= \varphi(x,t), & (x,t) &\in S_h. \end{aligned} \tag{5.55}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$,

$$\Lambda_{(5.55)}(z(x,t)) \equiv \varepsilon^* \Lambda^2_{(5.55)} z(x,t) - p(x,t)\delta_{\bar{t}} z(x,t) - g(x,t,z(x,t)),$$

$$\Lambda^*_{(5.55)} \equiv \sum_{s=1,2} a_s(x,t)\delta_{\overline{x_s}\,\widehat{x_s}} + \sum_{s=1,2} [\; b_s^+(x,t)\delta_{x_s} + b_s^-(x,t)\delta_{\overline{x_s}} \;] - c^0(x,t),$$

where b_s^+ and b_s^- are respectively the positive and the negative part of b_s. The approximation of normalised diffusive fluxes is constructed below.

Considering the difference scheme (5.55), (5.54), we assume that the estimates of Theorem 5.1 are fulfilled. Using the maximum principle [16], the estimate

$$|u(x,t) - z(x,t)| \le M\,[\; \varepsilon^{-1}N^{-1} + N_0^{-1} \;], \quad (x,t) \in \overline{G}_h. \tag{5.56}$$

can be proved.

To construct a ε-uniformly convergent difference scheme, we apply a special grid condensed in the boundary layer [22]. On the set $\overline{G}$ we introduce the grid

$$\overline{G}_{h\,(5.57)} = \overline{G}_{h\,(5.57)}(\sigma) = \overline{D}_{h\,(5.57)}(\sigma) \times \overline{\omega}_0, \tag{5.57}$$

where

$$\overline{D}_h(\sigma) = \overline{\omega}_1^* \times \overline{\omega}_2^*, \quad \overline{\omega}_s^* = \overline{\omega}_s^*(\sigma) = \overline{\omega}_s^*(\sigma, d_s), \quad s = 1, 2,$$

the grid $\overline{\omega}^*_{s\,(5.57)}$ is the grid $\overline{\omega}^*_{1\,(3.43)}$ with d and N equal to d_s and N_s respectively. For this grid, the following estimate is valid

$$|u(x,t) - z(x,t)| \le M\,[\; N^{-1}\ln N + N_0^{-1} \;], \; (x,t) \in \overline{G}_{h(5.57)}. \tag{5.58}$$

We now construct the approximation of the normalised fluxes $P_1(x,t)$, $P_2(x,t)$ for the special difference scheme (5.55), (5.57). For this purpose we need to modify the standard difference derivatives with respect to variables x_1 and x_2. Let the estimate

$$|u(x,t) - z(x,t)| \le \beta(N, N_0), \quad (x,t) \in \overline{G}_{h\,(5.57)}, \tag{5.59a}$$

hold, where $\beta(N, N_0)$ tends to zero ε-uniformly for N, $N_0 \to \infty$. The computational parameter h_s^* is defined by the relation

$$h_s^* = h_s^*(\varepsilon, \beta(N, N_0)) = \min\,[\; 4^{-1}d_s,\; M\varepsilon\beta^{1/2}(N, N_0) \;], \tag{5.59b}$$

where $M = M_{(5.59)}$ is an arbitrary number. We introduce grid sets $\overline{G}_h^{1-}$, $\overline{G}_h^{2-}$

$$\overline{G}_h^{s-} = \overline{G}_{h\,(5.57)} \cap \{(x,t):\; x_s \le d_s - h_s^*\}, \quad s = 1, 2.$$

By using linear interpolation along x_s for the grid function $z(x,t)$ we construct functions $\widetilde{z}^s(x,t)$ which are continuous functions along x_s and grid functions along variables t, x_{3-s}, $s = 1, 2$. Then we form modified difference derivatives

$$\delta^*_{x1} z(x,t) = (h_1^*)^{-1}[\; \widetilde{z}^1(x_1 + h_1^*, x_2, t) - z(x,t) \;], \quad (x,t) \in \overline{G}_h^{1-},$$

$$\delta^*_{x2} z(x,t) = (h_2^*)^{-1}[\; \widetilde{z}^2(x_1, x_2 + h_2^*, t) - z(x,t) \;], \quad (x,t) \in \overline{G}_h^{2-}.$$

We emphasise that in order to construct the modified difference derivatives we use the function $\beta(N, N_0)$, that is the right-hand side in the estimate (5.59a) . In this case the function $\beta(N, N_0)$ can be taken to be the right-hand side in the inequality (5.58).

The normalised diffusive fluxes $P_1(x,t)$, $P_2(x,t)$ are approximated by the grid functions $P_1^{*\,h+}(x,t)$, $P_2^{*\,h+}(x,t)$, where

$$P_s^{*\,h+}(x,t) = \varepsilon\delta_{xs}^* z(x,t), \quad (x,t) \in \overline{G}_h^{s-}, \ s = 1,2.$$

Using the estimate (5.58) we establish ε-uniform convergence of the functions $P_1^{*\,h+}(x,t)$, $P_2^{*\,h+}(x,t)$, that is the computed normalised fluxes,

$$\begin{gathered} |P_s(x,t) - P_s^{*\,h+}(x,t)| \le M\,[\, N^{-1}\ln N + N_0 \,]^{1/2}, \\ (x,t) \in \overline{G}_h^{s-}, \quad s = 1,2. \end{gathered} \tag{5.60}$$

Theorem 5.1 *Let a_s, b_s, c^0, $p \in C^{l+\alpha}(\overline{G})$, $s = 1,2$, $g \in C^{l+\alpha}(\overline{G} \times R)$, $\varphi \in C^{l+\alpha}(S)$, $U \in C^{l-2+\alpha}(\overline{G})$ (where $U(x,t)$ is the regular part of the solution of the boundary value problem (5.52)), $l \gt 6$, $\alpha > 0$. Then the solution of the difference scheme (5.55), (5.57) and the computed diffusive fluxes $P_s^{*\,h+}(x,t)$, $(x,t) \in \overline{G}_h^{s-}$, $s = 1,2$ converge ε-uniformly. For the solution of the difference scheme and the computed fluxes $P_1^{*\,h+}(x,t)$, $P_2^{*\,h+}(x,t)$ the estimates (5.56), (5.58), (5.60) hold.*

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(To be continued)